ESCI 343 – Atmospheric Dynamics II
Lesson 2 – Q-G Height-tendency Equation

Reference: *An Introduction to Dynamic Meteorology* (3rd edition), J.R. Holton
Synoptic-dynamic Meteorology in Midlatitudes, Vol 1, H.B. Bluestein

THE QUASIGEOSTROPHIC THERMODYNAMIC ENERGY EQUATION

- The thermodynamic energy equation in pressure coordinates is

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J \,, \qquad (1)$$

which when expanded out, and using the definition of ω, becomes

$$\underset{A}{\frac{\partial T}{\partial t}} = \underset{B}{-\vec{V} \bullet \nabla_p T} - \left(\underset{C}{\frac{\partial T}{\partial p}} - \underset{D}{\frac{\alpha}{c_p}} \right)\omega + \underset{E}{\frac{J}{c_p}} \,. \qquad (2)$$

In this form, the terms represent:

Term A – Local temperature tendency

Term B – Horizontal thermal advection

Term C – Vertical thermal advection

Term D – Adiabatic expansion/compression due to vertical motion

Term E – Diabatic heating (radiation, latent heat, etc.)

- Terms *C* and *D* can be combined and written as

$$\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} = \left(\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p} \right) \frac{p}{R_d} \,,$$

and defining the *static-stability parameter*, σ, as

$$\sigma \equiv -\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p} \,, \qquad (3)$$

we get the following form of the thermodynamic energy equation in pressure coordinates.

$$\underset{A}{\frac{\partial T}{\partial t}} = \underset{B}{-\vec{V} \bullet \nabla_p T} + \underset{C}{\frac{\sigma p}{R_d}\omega} + \underset{D}{\frac{J}{c_p}} \,. \qquad (4)$$

- In this form of the equation, the vertical advection and adiabatic expansion/compression are combined into one term, Term C.

- The static stability parameter is a positive number for a stable atmosphere, and a negative number for an unstable atmosphere.
- The quasigeostrophic form of the thermodynamic energy equation in pressure coordinates is

$$\left(\frac{\partial}{\partial t}+\vec{V}_g \bullet \nabla_p\right)T-\left(\frac{\sigma p}{R_d}\right)\omega=\frac{J}{c_p} \tag{5}$$

where we have simply substituted the geostrophic wind for the actual wind in the advection term.

THE HYDROSTATIC EQUATION IN PRESSURE COORDINATES

- The hydrostatic equation in pressure coordinates is derived as follows:

In height coordinates we have

$$\frac{\partial p}{\partial z}=-\rho g\ . \tag{6}$$

Using the chain rule

$$\frac{\partial p}{\partial z}=\frac{\partial p}{\partial \Phi}\frac{\partial \Phi}{\partial z}=g\frac{\partial p}{\partial \Phi},$$

and so from (6)

$$g\frac{\partial p}{\partial \Phi}=-\rho g$$

or

$$\frac{\partial \Phi}{\partial p}=-\alpha\ . \tag{7}$$

- Equation (7) is the hydrostatic equation in pressure coordinates.

AN ALTERNATE FORM OF THE QG THERMODYNAMIC EQUATION

- From the ideal gas law we can write equation (7) as

$$\frac{\partial \Phi}{\partial p}=-\alpha=-\frac{R_d T}{p}, \tag{8}$$

which when solved for T gives

$$T = -\frac{p}{R_d}\frac{\partial \Phi}{\partial p}. \quad (9)$$

- Substituting (9) for the temperature in the QG thermodynamic energy equation, (5) gives

$$\left(\frac{\partial}{\partial t} + \vec{V}_g \bullet \nabla_p\right)\frac{\partial \Phi}{\partial p} + \sigma\omega = -\frac{J}{c_p}\frac{R_d}{p}, \quad (10)$$

which rearranged yields

$$\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial t} = -\vec{V}_g \bullet \nabla_p\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{R_d J}{p c_p}. \quad (11)$$

- Geopotential tendency is defined as

$$\chi = \frac{\partial \Phi}{\partial t}, \quad (12)$$

so the QG thermodynamic energy equation becomes

$$\frac{\partial \chi}{\partial p} = -\vec{V}_g \bullet \nabla_p\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{R_d J}{p c_p}. \quad (13)$$

- **NOTE! Equations (5) and (13) are identical! They are just written in different forms.**

THE QG VORTICITY EQUATION REVISITED

- The QG vorticity equation in pressure coordinates is

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \bullet \nabla_p\left(\zeta_g + f\right) + f_0\frac{\partial \omega}{\partial p}. \quad (14)$$

- The geostrophic vorticity in terms of the geopotential is

$$\zeta_g = \frac{1}{f_0}\nabla_p^2\Phi. \quad (15)$$

- Substituting (15) into (14) give another form of the QG vorticity equation

$$\nabla^2\chi = -f_0\vec{V}_g \bullet \nabla_p\left(\frac{1}{f_0}\nabla_p^2\Phi + f\right) + f_0^2\frac{\partial \omega}{\partial p}. \quad (16)$$

- **NOTE! Equations (14) and (16) are identical! They are just written in different forms.**

- Equations (13) and (16) are two equations with two dependent variables, χ and ω.
  - If we know the what the geopotential field (Φ) is, then these equations form a complete system which can be solved for either χ or ω.

THE GEOPOTENTIAL TENDENCY EQUATION

- The first equation we will derive is the geopotential tendency equation, found by eliminating ω between (13) and (16).
- The idea behind this is simple, but the individual mathematical steps become complicated.
  - Differentiate (13) with respect to pressure, then multiply it by f_0^2/σ.
  - Add the result to (16).
- The result is (note that for ease of notation the subscript p is no longer written on the del operator, but is implied).

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\chi = -f_0\vec{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right] - \frac{f_0^2 R_d}{\sigma c_p}\frac{\partial}{\partial p}\left(\frac{J}{p}\right) - \frac{f_0^2}{\sigma}\omega\frac{\partial\sigma}{\partial p} \; . \tag{17}$$

- The static stability parameter normally increases with height; however, analysis of the Q-G tendency equation is slightly easier if we assume that σ is constant so that the last term disappears. In this case, the equation becomes

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\chi = -f_0\vec{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right] - \frac{f_0^2 R_d}{\sigma c_p}\frac{\partial}{\partial p}\left(\frac{J}{p}\right). \tag{18}$$

Q-G geopotential tendency equation

- Though ugly, this equation has a sort of inner beauty. We first try to see this beauty by analyzing the terms of the equation in a *qualitative* fashion.
- We do this by imaging that the horizontal structure of disturbances in the atmosphere can be approximated by sinusoidal functions such as

$$\chi(x, y, p, t) = \mathrm{X}(p,t)\exp[i(kx + ly)]\,. \tag{19}$$

- If we ignore the pressure derivatives on the left hand side (LHS) of the tendency equation, then the LHS becomes

$$\nabla^2\left[\chi(x,y,p,t)\right]=\nabla^2\left\{X(p,t)\exp[i(kx+ly)]\right\}$$
$$=-(k^2+l^2)X(p,t)\exp[i(kx+ly)]=-(k^2+l^2)\chi(x,y,p,t)$$

or more simply,

$$\nabla^2\chi\propto-\chi\,. \tag{20}$$

- What this means is that ***for a sinusoidal disturbance having a zero mean value, the horizontal Laplacian of a field is proportional to the negative of the field***.
- So, we can qualitatively think of the LHS of the equation as being nothing more than $-\chi$, so that ***if the LHS is negative it means that the geopotential tendency is positive.***
- It may help to write the equation using the 'proportional to' symbol ($\propto$), in the following manner,

$$-\chi \quad \propto \quad -f_0\vec{V}_g\bullet\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\bullet\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]-\frac{f_0^2R_d}{\sigma c_p}\frac{\partial}{\partial p}\left(\frac{J}{p}\right). \tag{21}$$

- If we can find the signs of the terms on the right hand side (RHS) of the equation we will be able to tell whether heights are going to rise or fall.

THE VORTICITY ADVECTION TERM

- Though the terms on the RHS look intimidating, they really aren't. The first term on the RHS is nothing more than absolute vorticity advection,

$$\textit{Absolute vorticity advection}=-\vec{V}_g\bullet\nabla\left(\zeta_g+f\right)=-\vec{V}_g\bullet\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right). \tag{22}$$

- If vorticity advection is positive, this means that the geopotential tendency, χ, is negative (falling heights).

THE DIFFERENTIAL THERMAL ADVECTION TERM

- Remember that we earlier found that

$$T=-\frac{p}{R}\frac{\partial\Phi}{\partial p}\,. \tag{23}$$

- This means that the second term on the RHS is proportional to the vertical derivative of temperature advection.
- If temperature advection decreases with height (increases with pressure) then

$$\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0$$

and so χ will be positive, so heights will be rising.

- Two important points to note:
    - *Remember that we are using pressure coordinates, so if something is increasing with height, it is decreasing with pressure, and therefore* $\partial/\partial p < 0$.
    - *It is the vertical derivative of the advection that matters.* Strong cold advection over weak cold advection has the same effect as weak warm advection over strong warm advection, because in both cases the derivative has the same value.

THE DIFFERENTIAL DIABATIC HEATING TERM

- The differential heating term (third term on RHS) behaves similarly to the differential thermal advection term.
    - If the heating decreases with height, or cooling increases with height, then heights will rise.
- Another useful way of writing the essence of the Q-G tendency equation is

$\partial\Phi/\partial t \propto$ −absolute vorticity advection +$\partial/\partial p$(thermal advection) +$\partial/\partial p$(heating)
or
$\partial\Phi/\partial t \propto$ −absolute vorticity advection −$\partial/\partial z$(thermal advection) −$\partial/\partial z$(heating)

- The previous analysis leads us to a very important conclusion. ***In quasi-geostrophic theory, there are only three ways for heights to fall…either through positive vorticity advection, through warm advection that increases with height, or through diabatic heating that increases with height!***

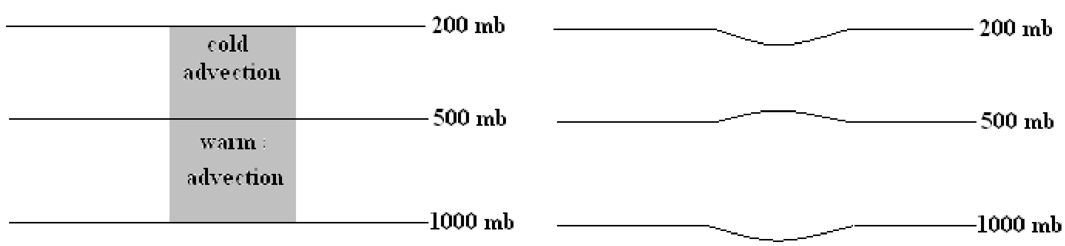
A PHYSICAL INTERPRETATION OF THE TENDENCY EQUATION

- The effects of the terms on the RHS of the tendency equation can be explained physically, as well as mathematically.
- *Vorticity advection*: We know that on the synoptic scale there is a direct relationship between vorticity and geopotential, via

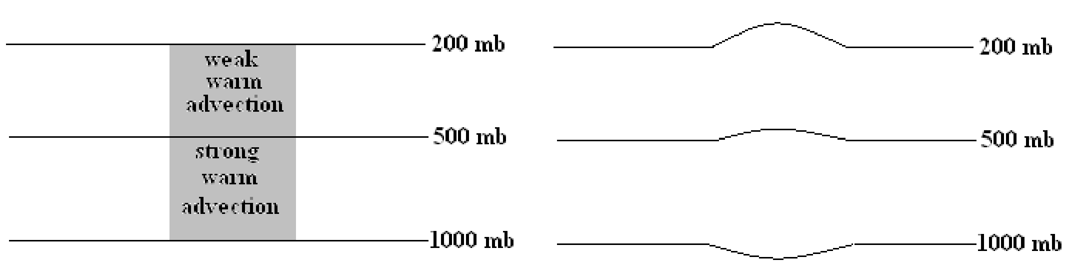
$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi \,. \tag{24}$$

- PVA leads to increasing values of vorticity, which if the atmosphere is going to remain in near geostrophic balance requires $\nabla^2\Phi$ must also increase, which means that Φ itself must adjust to a lower value (high values of $\nabla^2\Phi$ imply low values of Φ itself).
- In essence, the height anomaly is advected with the mean flow.

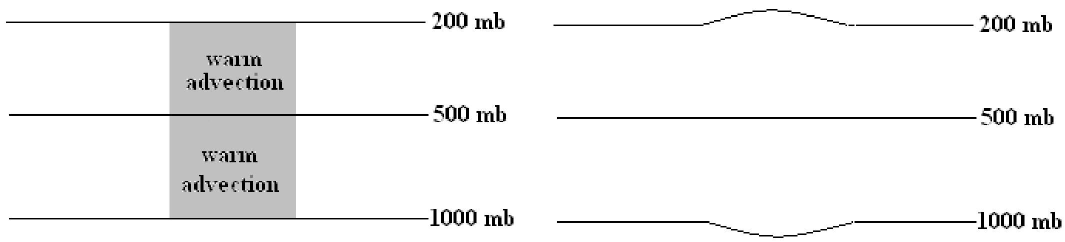
- *Differential thermal advection*: The effects of differential thermal advection can be thought of as follows:
  - Imagine a scenario where there is net warm advection in the lower levels (below 500 mb), and net cold advection in the upper levels (above 500 mb).
  - Since the thickness between two pressure surfaces is proportional to temperature, the low-level warm advection will lead to increased thickness of the 1000 – 500 mb layer, while the upper-level cold advection will lead to decreased thickness of the 500 – 200 mb layer.
  - The net result is height rises at 500 mb (see diagram below).



- The same result will occur with weak warm advection aloft and stronger warm advection in the low-levels (see diagram below).



- If the advection is the same strength both aloft and below, then there is no change in height at 500 mb (see diagram below).



- Typically, thermal advection is very small in the upper troposphere (above 500 mb) compared to that in the lower-levels, so it is really the low level advection that determines the 500 mb geopotential tendency.
  - Cold advection in the lower levels will decrease the thickness of the 1000 –500 mb layer, and lower the heights at 500 mb, as would be expected (since *cold advection decreasing with height is the same as warm advection increasing with height*).
  - Warm advection in the lower levels will increase the thickness of the 1000 –500 mb layer, and result in height rises at 500 mb.
- *Differential diabatic heating*. The physical interpretation for the differential diabatic heating term is similar to that for differential advection.
  - If there is more heating above a level than below it, the heights at that level will fall.
    - Phrased another way, we can say that above the level of maximum heating (J/p) the heights will rise, and below the level of maximum heating heights will fall.

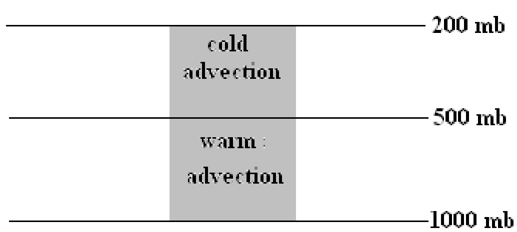
Le CHATELIER'S PRINCIPLE

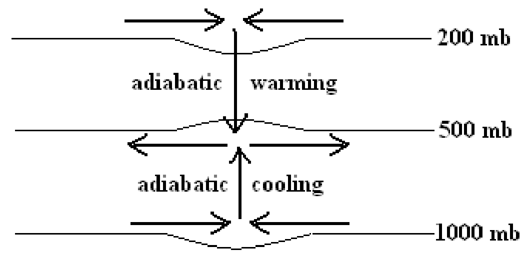
- Le Chatelier's Principle, named for Henry Louis Le Chatelier, states that many natural systems will resist changes, and if forced to change, will react with process that try to restore the original state.
- Though Le Chatelier's Principle isn't as rigorous and general as often thought to be[1], we can see Le Chatelier's principle at work in the differential thermal advection and diabatic heating terms of the Q-G tendency equation.

---

[1] see J. de Heer, *J. Chem. Educ.*, **34**, 375 (1957)

- For example, cold advection (or diabatic cooling) over warm advection (or diabatic heating) forces height rises at 500 mb, as well as height falls at 200 and 1000 mb (as per the diagram below.)





- However, these height rises and falls indicate that there must be a change in the vorticity at these levels (increased vorticity where there are height falls, and decreased vorticity where there are height rises.)
- To accomplish this vorticity change in a quasi-geostrophic framework, there must be convergence where there are height falls, and divergence where there are height rises.
- This convergence/divergence pattern is the result of the isallobaric wind.
- The convergence/divergence pattern leads to upward motion and adiabatic cooling in the lower levels, and subsidence and adiabatic warming in the upper levels.
- The adiabatic heating/cooling opposes the original temperature change due to advection.
- LeChatelier's Principle doesn't mean that the effects of the differential heating (advection) will be completely cancelled by the adiabatic heating/cooling from the secondary circulation, but does illustrate that the atmosphere will resist the changes imposed by the thermal forcing, and will respond with a secondary circulation.

* LeChatelier's Principle can also be seen in the vorticity advection term of the QG height tendency equation.
  - PVA leads to falling heights. But, the advective wind is divergent in regions of PVA, and since divergence decreases vorticity this effect is counter to the vorticity increase due to the PVA.

EFFECTS OF STATIC STABILITY

- The static stability of the troposphere on the synoptic scale is rarely negative.
- Since static stability appears in the denominator of the heating terms, an increase in static stability will cause the height rises or falls from these terms to be of a less magnitude than in a less stable atmosphere.
- The vertical change of static stability is a little more complex. The static stability term (which we've previously neglected) has the following effect on the height tendency:

$$\chi \propto \frac{f_0^2}{\sigma} \omega \frac{\partial \sigma}{\partial p}. \tag{25}$$

- Since static stability usually increases with height, this implies that downward vertical motion will lead to a lowering of the geopotential heights, while upward motion will lead to raising of geopotential heights. To understand this physically, recall the from the thermodynamic energy equation that vertical motion affects temperature tendency via

$$\frac{\partial T}{\partial t} \propto \sigma \omega. \tag{26}$$

- The impact of the vertical motion is enhanced as the static stability increases. Therefore, if the motion is downward, there will be more heating at higher altitudes (where σ is larger) than at lower altitudes (where σ is smaller). Thus, there will be heating increasing with height, which we have already seen leads to height falls.
- For upward motion, there will be more cooling aloft than below, which will lead to height rises.
- If static stability decreases with height, then the effect of vertical motion is opposite from that just described, since there will be larger heating (or cooling) below, rather than aloft.

AN ADVANCED TREATMENT OF THE TENDENCY EQUATION

- Since we've previously assumed that disturbances in the atmosphere have a sinusoidal horizontal structure, we will also assume that the forcing (terms on the RHS of the tendency equation) also have a sinusoidal structure. So, we assume

$$\chi(x, y, p, t) = \mathrm{X}(p,t)\exp[-(kx + ly)]$$
$$vorticity\ advection\ term = F_v(p)\exp[-(kx + ly)]$$
$$thermal\ advection\ term = -\frac{dF_T(p)}{dp}\exp[-(kx + ly)]$$
$$diabatic\ heating\ term = -\frac{dF_J(p)}{dp}\exp[-(kx + ly)]$$

and put these into the tendency equation. This gives an ordinary differential equation,

$$\frac{d^2\mathrm{X}}{dp^2} - \alpha^2\mathrm{X} = \frac{\sigma}{f_0^2}\left(F_v - \frac{dF_T}{dp} - \frac{dF_J}{dp}\right) \tag{27}$$

where

$$\alpha^2 = \frac{K^2\sigma}{f_0^2} \tag{28}$$

(we've assumed the static stability parameter, σ, is constant with height).

- The solutions to (27) are hyperbolic sines and cosines with a characteristic vertical length scale of

$$D = \frac{2\pi}{\alpha} = \frac{2\pi f_0}{K\sqrt{\sigma}} = \frac{f_0 L}{\sqrt{\sigma}} \tag{29}$$

where L is the wavelength of the disturbance.

- So, *the longer the wavelength of a disturbance, the deeper the effects of its forcing terms are felt in the atmosphere*.

QUASI-GEOSTROPHIC POTENTIAL VORTICITY

- The tendency equation (ignoring the diabatic heating term, J) can be written as

$$\frac{D_g}{Dt}\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right)\right] = 0\,.$$

- The quantity in brackets is called the *quasi-geostrophic potential vorticity*,

$$q \equiv \frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)$$

and is conserved following a fluid parcel in adiabatic motion.

EXERCISES

1. Derive the Q-G tendency equation, showing all steps.

2. Is the vertical extent of the forcing terms in the Q-G tendency equation larger or smaller in the tropics as compared to the middle latitudes?