

ESCI 342 – Atmospheric Dynamics I

Lesson 13 – The Vorticity Equation

Reference: *An Introduction to Dynamic Meteorology* (4th edition), Holton
Atmosphere-Ocean Dynamics, Gill

Reading: Martin, Section 5.3

DERIVATION OF THE VORTICITY EQUATION

- An equation for the change in vorticity with time can be derived from the horizontal momentum equations,

$$\frac{\partial u}{\partial t} + \vec{V} \cdot \nabla u = -\alpha \frac{\partial p}{\partial x} + fv \quad (1)$$

$$\frac{\partial v}{\partial t} + \vec{V} \cdot \nabla v = -\alpha \frac{\partial p}{\partial y} - fu. \quad (2)$$

- Taking $\partial/\partial x$ of (2) and subtracting $\partial/\partial y$ of (1) gives

$$\frac{\partial}{\partial x} \frac{\partial v}{\partial t} - \frac{\partial}{\partial y} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (\vec{V} \cdot \nabla v) - \frac{\partial}{\partial y} (\vec{V} \cdot \nabla u) = -\frac{\partial}{\partial x} \left(\alpha \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial y} \left(\alpha \frac{\partial p}{\partial x} \right) - \frac{\partial}{\partial x} (fu) - \frac{\partial}{\partial y} (fv)$$

which can be expanded and rearranged to

$$\frac{\partial}{\partial t} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial x} (\vec{V} \cdot \nabla v) - \frac{\partial}{\partial y} (\vec{V} \cdot \nabla u) = - \left(\frac{\partial \alpha}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y} \frac{\partial p}{\partial x} \right) - f \nabla_H \cdot \vec{V} - \beta v$$

which then rearranges as

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} (\vec{V} \cdot \nabla v) - \frac{\partial}{\partial y} (\vec{V} \cdot \nabla u) = - \left(\frac{\partial \alpha}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y} \frac{\partial p}{\partial x} \right) - f \nabla_H \cdot \vec{V} - \beta v.$$

- It turns out that

$$\frac{\partial}{\partial x} (\vec{V} \cdot \nabla v) - \frac{\partial}{\partial y} (\vec{V} \cdot \nabla u) = \vec{V} \cdot \nabla \zeta + \zeta \nabla_H \cdot \vec{V} + \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right)$$

so that the equation for vorticity can be written as

$$\frac{\partial \zeta}{\partial t} = -\vec{V} \cdot \nabla \zeta - \beta v - \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) - \left(\frac{\partial \alpha}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y} \frac{\partial p}{\partial x} \right) - (\zeta + f) \nabla_H \cdot \vec{V}.$$

or as

$$\frac{\partial \zeta}{\partial t} = \underbrace{-\vec{V} \cdot \nabla \zeta}_A - \underbrace{\beta v}_B + \underbrace{\hat{k} \cdot \frac{\partial \vec{V}}{\partial z} \times \nabla w}_D - \underbrace{\hat{k} \cdot \nabla \alpha \times \nabla p}_E - \underbrace{\eta \nabla_H \cdot \vec{V}}_F. \quad (3)$$

PHYSICAL MEANING OF THE TERMS IN THE VORTICITY EQUATION

- Term A:** The local relative vorticity tendency
- Term B:** Advection of relative vorticity
- Term C:** Advection of planetary vorticity
 - Accounts for generation of relative vorticity due to movement of the air poleward.
 - Imagine the air at rest with respect to the Earth so there is zero relative vorticity. As the air moves toward the south, the local rotation of the Earth has decreased, so the air appears to have taken on a cyclonic circulation, even though from space,

its rotation hasn't changed. Thus, southward motion leads to an increase in relative vorticity, while northward motion leads to a decrease.

- o Since $-\beta v = -\vec{V} \cdot \nabla f$, this term appears as an advection of planetary vorticity, and that's what it is commonly called. It is also often referred to as simply the "beta term".
- **Term D:** Twisting/tilting term
 - o Accounts for the tilting of horizontal relative vorticity into the vertical.
- **Term E:** Solenoidal term
 - o Accounts for the generation of relative vorticity due to baroclinicity.
- **Term F:** Divergence term
 - o Accounts for the generation of relative vorticity due to convergence.
 - o The physical explanation of this term is simply the conservation of absolute angular momentum (angular momentum as viewed from space).
 - o If a circulation has positive absolute angular momentum (and therefore positive absolute vorticity), if it converges, it must spin faster in the positive direction, and will gain cyclonic absolute vorticity.
 - o If a circulation has negative absolute angular momentum (and therefore negative absolute vorticity), if it converges, it must spin faster in the negative direction and will gain anticyclonic absolute vorticity.

THE VORTICITY EQUATION ON THE SYNOPTIC SCALE

- On the synoptic scale, scale analysis shows that the vertical advection term, the solenoidal term, and the twisting/tilting term are all an order of magnitude less than the next largest terms. Therefore, we can ignore these terms and write the vorticity equation (for the synoptic scale only) as

$$\frac{\partial \zeta}{\partial t} = -\vec{V} \cdot \nabla_H \zeta - \beta v - \eta \nabla_H \cdot \vec{V} \quad (4)$$

- Note: On the synoptic scale $\zeta \ll f$, so that $\eta \cong f$. Therefore, most authors write the vorticity equation with only planetary vorticity, rather than absolute vorticity, in the divergence term, as

$$\frac{\partial \zeta}{\partial t} = -\vec{V} \cdot \nabla_H \zeta - \beta v - f \nabla_H \cdot \vec{V} .$$

- o This is the form of the vorticity equation we will usually use, though you should always keep in mind that it is the absolute vorticity, not the planetary vorticity, that is really in this term.
- Because f is only dependent on y , equation (4) can be written as

$$\frac{D}{Dt} (\zeta + f) = -f \nabla_H \cdot \vec{V}$$

or

$$\frac{D\eta}{Dt} = -f \nabla_H \cdot \vec{V} . \quad (5)$$

- Equation (5) is one of the most important equations in all of meteorology. I expect you to memorize it, and to understand what it means.
- Equation

- (5) states that *on the synoptic scale the absolute vorticity of a fluid parcel changes mainly in response to divergence or convergence.*
 - Divergence leads to a decrease in absolute vorticity.
 - Convergence leads to an increase in absolute vorticity.
- **Caution:** By only including the planetary vorticity in the divergence term, we've constrained convergence to always produce cyclonic vorticity. This is okay on the synoptic scale, because negative absolute vorticity rarely if ever occurs at that scale.
 - Keep in mind though, that on the mesoscale or smaller, the absolute vorticity must be used in the divergence term (we can't ignore ζ), and so it is possible on smaller scales for convergence to lead to creation of negative vorticity.

THE VORTICITY EQUATION AND THE ROLE OF CONSERVATION OF ANGULAR MOMENTUM

- We've already mentioned that the convergence/divergence term is based on conservation of absolute angular momentum. This may not be completely clear from our derivation of the vorticity equation. The question may be asked, "Can the vorticity equation be derived directly from conservation of absolute angular momentum?" The answer is "yes it can." If you are interested in seeing this derivation, go to www.atmos.millersville.edu/~adecaria/DERIVATIONS/Vorticity.pdf.

THE QUASI-GEOSTROPHIC APPROXIMATION

- Another assumption that can be made on the synoptic scale is that the actual wind can be approximated by the geostrophic wind ($\vec{V} \cong \vec{V}_g$) in every term except the divergence term.
 - This is the *quasi-geostrophic* approximation.
 - The rationale for this approximation is that the ageostrophic wind is usually much smaller than the geostrophic wind and can therefore be ignored in the advection terms.

$$\vec{V} = \vec{V}_g + \vec{V}_a$$

$$\vec{V}_a \ll \vec{V}_g$$

- However, since the ageostrophic wind is the only component of the wind that can be divergent, it must be retained in the divergence term.
- Using the quasi-geostrophic approximation, equation (4) becomes

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla_H \zeta_g - \beta v_g - f \nabla_H \cdot \vec{V}_a. \quad (6)$$

- **Equation (6) is the quasi-geostrophic vorticity equation.**
- ζ_g is the geostrophic vorticity (the vorticity of the geostrophic wind). It can be written as

$$\zeta_g = \nabla^2 \psi. \quad (7)$$

- If there is no divergence, then (5) says that the absolute vorticity must be conserved following a fluid parcel,

$$\frac{D\eta}{Dt} = 0. \quad (8)$$

- Thus, *at the level-of-nondivergence, vorticity changes only due to advection, and absolute vorticity is conserved following the fluid parcel.*

VORTICITY ADVECTION AND THE MOVEMENT OF SYNOPTIC DISTURBANCES

- Focusing on the advection terms of the Q-G vorticity equation, and ignoring the effects of convergence and divergence, we get

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla_H \zeta_g - \beta v_g. \quad (9)$$

- The first term on the RHS represents advection of relative vorticity, and the second term represents planetary vorticity.
- For a typical midlatitude disturbance these terms tend to have opposite effects.
 - The relative vorticity advections favors eastward motion
 - The planetary vorticity advection term favors westward (retrograde) motion.
- Which term “wins” depends on the size, or wavelength of the disturbance.
 - For very large disturbances the planetary term will dominate, and the wave will propagate westward.
 - For smaller disturbances the relative vorticity advection is more significant, and the disturbance will move eastward.

EXERCISES

1. Expand $\frac{D}{Dt}(\zeta + f)$ to show that it equals $\frac{\partial \zeta}{\partial t} + \vec{V} \cdot \nabla \zeta + \beta v$.

2. a. Show that $-\left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z}\right)$ is the same as $\hat{k} \cdot \frac{\partial \vec{V}}{\partial z} \times \nabla w$.

b. Show that $-\left(\frac{\partial \alpha}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y} \frac{\partial p}{\partial x}\right)$ is the same as $-\hat{k} \cdot \nabla \alpha \times \nabla p$.

3. a. Starting with the momentum equations in isobaric coordinates

$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial x} + fv \quad (a)$$

$$\frac{Dv}{Dt} = -\frac{\partial \Phi}{\partial y} - fu \quad (b)$$

take $\partial/\partial x$ of (b) and subtract $\partial/\partial y$ of (a) to derive the vorticity equation in isobaric coordinates,

$$\frac{\partial \zeta}{\partial t} = -\vec{V} \cdot \nabla \zeta - \beta v + \hat{k} \cdot \frac{\partial \vec{V}}{\partial p} \times \nabla \omega - (\zeta + f) \nabla_p \cdot \vec{V}.$$

b. Why is there no solenoidal term?

4. a. Remembering that the geostrophic wind can be written in terms of the streamfunction, ψ , as

$$\vec{V}_g = \hat{k} \times \nabla \psi$$

show that $\zeta_g = \nabla^2 \psi$.

b. Show that $\vec{V}_g \cdot \nabla \zeta_g = J(\psi, \nabla^2 \psi)$ where $J(a, b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x}$. ($J(a, b)$ is called the Jacobian of a and b .)

c. Show that the quasi-geostrophic vorticity equation can be written as

$$\frac{\partial}{\partial t} (\nabla^2 \psi) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = -f \nabla \cdot \vec{V}_a$$

5. Show that $\frac{\partial}{\partial x} \frac{D}{Dt} \neq \frac{D}{Dt} \frac{\partial}{\partial x}$ (i.e., material derivatives don't commute with partial derivatives).