TURBULENT KINETIC ENERGY

- Turbulent kinetic energy (TKE) is the kinetic energy (per unit mass) associated with turbulent eddies.
- Mathematically it is defined as
  \[ e = \frac{1}{2} \left( u'^2 + v'^2 + w'^2 \right). \]  
- Turbulent kinetic energy is primarily transferred from larger scales to smaller scales.
  - At the large scales the turbulence is generated via mechanical means (through sheared flows) or via buoyancy.
  - Energy is *dissipated* at the very smallest scales due to molecular friction. This flow of turbulent energy from large scales to small scales is known as the *energy cascade*.
  - The energy cascade is summed up very aptly in the verse,

  \[ \text{Big whorls have little whorls,} \]
  \[ \text{Which feed on their velocity,} \]
  \[ \text{And little whorls have lesser whorls,} \]
  \[ \text{And so on to viscosity.} \]

  - *L. F. Richardson*

- An equation for the budget of turbulent kinetic energy (TKE) is given symbolically as
  \[ \frac{\overline{De}}{Dt} = MP + BP + TR - \varepsilon. \]  

References:
- *An Introduction to Dynamic Meteorology (3rd edition)*, J.R. Holton
- *An Introduction to Boundary Layer Meteorology*, R.B. Stull
- *Structure of the Atmospheric Boundary Layer*, Sorbjan
o $MP$ represents the mechanical production of turbulence.

o $BP$ represents buoyant production.

o $TR$ represents turbulent transport and redistribution by pressure perturbations.

o $\varepsilon$ represents molecular dissipation.

- **Mechanical production (MP)** – The mechanical production term has the form

$$MP = -u w \frac{\partial u}{\partial z},$$

(3)

Mechanical production occurs because of dynamic instabilities caused by sheared flow. The stronger the shear of the mean flow, the more turbulence will be produced.

This term can also be negative, and represent a loss of TKE.

- **Buoyant production (BP)** – The buoyant production term has the form

$$BP = w b',$$

(4)

where

$$b' = -g \frac{\rho'}{\rho_0}$$

(5)

and is the buoyancy perturbation.

Buoyant production represents the production of turbulence due to thermals caused by heating of the surface (in the ocean, buoyant production would result from cooling of the surface).

Like MP, the BP term is usually largest near the surface. In a stable layer this term will be negative and represent a loss of TKE.

- **Transport and redistribution (TR)** – This term has two components, and is written as

$$TR = -\left( u', \frac{\partial e}{\partial x} + w', \frac{\partial e}{\partial z} \right) - \frac{1}{\rho_0} \left( u', \frac{\partial p'}{\partial x} + w', \frac{\partial p'}{\partial z} \right).$$

(6)

The first component represents the transport (or advection) of TKE by turbulent eddies.

The second component represents the effects of pressure perturbations.
- **Dissipation** – The dissipation term has the form

\[ \varepsilon = \nu \left[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right) (u' + w') \right]^2. \]

- Dissipation represents the loss of energy due to molecular friction (\( \nu \) is the kinematic viscosity), and is always positive, so that in the TKE equation dissipation always represents a loss term.

**OTHER TURBULENCE CHARACTERIZATIONS**

- **Velocity scale**
  - One possible velocity scale is

\[ u_m^2 = \overline{u'^2}. \]  

(8)

- If the turbulence is isotropic, then \( w'' \) is of the same order as \( u'' \), and Eq. (8) can be written as

\[ u_m^2 = \overline{u w} \]

(9)

(absolute value is needed to ensure that the velocity is real and not imaginary. This gives rise to the definition of friction velocity

\[ u^* = \left( \overline{u w} \right)^{1/2}. \]

(10)

- **Eddy size**
  - Eddy size is determined statistically
  - Use a “two-point” correlation function

\[ R_w(r) = \overline{w'(x)w'(x+r)}. \]

(11)

- The distance at which the velocity is uncorrelated \((R = 0)\) is taken to be the characteristic eddy size or length scale, \( l \).

- The correlation function can also be defined in terms of horizontal velocities

\[ R_u(r) = \overline{u'(x)u'(x+r)}. \]

(12)
Fourier spectra

- The Fourier transform pairs for a function of \( x \) are

\[
F(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx) \, dx
\]

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(ikx) \, dk.
\]  

- The Fourier transform of the correlation function \( R_u(r) \) gives the spectral energy density of the turbulence

\[
\phi(k) = \int_{-\infty}^{\infty} R_u(r) \exp(-ikr) \, dr.
\]

- The wave number at which the energy density peaks \( (k_p) \) can be taken as a characteristic wave number of the turbulence, and from the relation

\[
\lambda_p = 2\pi/k_p
\]

characteristic wavelength of the turbulent eddies can be computed.

Dissipation rate

- The rate at which turbulence dissipates kinetic energy is another means of characterizing turbulence.

- Dissipation rate is given the symbol \( \varepsilon \) and has units of \( \text{J s}^{-1} \text{kg}^{-1} \).

- Dissipation converts kinetic energy into thermal energy, due to viscous effects.

- Dissipation rate has been shown to be proportional to

\[
\varepsilon \propto u_m^3/l.
\]

Taylor’s Hypothesis

- The correlation function \( R(r) \) is a spatial correlation. However, data are usually taken at a single point as a time series.

- We make use of Taylor’s hypothesis in order to use the time series data to infer spatial properties of the turbulence.

- Taylor’s hypothesis states that if \( u'/\bar{u} \ll 1 \), then \( x = Ut \), where \( U \) is the mean velocity.
Using Taylor’s hypothesis we can do spectral analysis in frequency ($\omega$), and then convert it to wave number via

$$k = \omega/U.$$  

TURBULENCE SPECTRA

- The energy spectra of turbulence is theorized to have some general characteristics.
- The lowest wave numbers are the energy containing range.
- The highest wave numbers are the dissipation range
  - In the dissipation range, dimensional analysis predicts that the spectral density should be a function of dissipation rate and wave number only, via
    $$\phi(k) = a \epsilon^{2/3} k^{-2/3}. \quad (16)$$  
    - If the proportionality constant is known, and if the spectral density is measured, then the dissipation rate can be inferred.
- In between the energy containing range and the dissipation range is the inertial subrange, where the energy is “just passing through” on its way to smaller scales.
  - In the inertial subrange, dimensional analysis predicts that the spectral density should be given by
    $$\phi(k) = b \epsilon^{2/3} k^{-5/3}. \quad (17)$$
- On a log-log plot, an idealized turbulence spectrum would look like