INDEX OF REFRACTION

- The index of refraction (sometimes called the refractive index) is a complex number, \( m = m_r + im_i \).
  - The real part \( m_r \) is what we usually work with, and is the ratio of the speed of light in a vacuum \( c_{\text{vac}} \) to the speed of light in the medium in question \( c' \),
    \[
    m_r = \frac{c_{\text{vac}}}{c'}.
    \]
  - The imaginary part \( m_i \) is related to the absorption of radiation by the medium, and can be related to the absorption or mass-absorption cross sections via
    \[
    k_\lambda \rho = \sigma_\lambda n = \frac{4\pi m_i}{\lambda}.
    \]
- When light passes from one medium to another, the index of refraction changes, and hence, so does the speed of light.
- A ray is defined as a line perpendicular to the wave front.
- As the light passes from one medium into another, there is both reflection and refraction.
  - The wave fronts bend as they cross from one medium into another. Therefore, a ray will bend as it crosses from one medium into another.
  - The amount a ray is bent is given by Snell’s Law, which states that
    \[
    \frac{\sin \theta_1}{\sin \theta_2} = \frac{m_{r2}}{m_{r1}}
    \]
    where \( \theta_1 \) is the angle of incidence (and reflection), and \( \theta_2 \) is the angle of refraction (see diagram below).
INDEX OF REFRACTION FOR AIR

- The index of refraction for visible light in air depends on both the wavelength of the light and on the temperature of the air.
- The real part of the index of refraction for visible light in air decreases with increasing temperature.
  - The diagram below shows $m_r - 1$ as a function of temperature for yellow light.
  - Though the change in index of refraction with temperature is slight, it is enough to be important. Therefore
  - Light travels faster in warmer air, and slower in colder air.
  - Because the index of refraction is larger in colder air, from Snell's law we deduce that light bends toward colder temperatures.
- The real part of the index of refraction for visible light in air decreases with increasing wavelength.
The diagram below shows $m_r - 1$ as a function of wavelength.

The change of the index of refraction in air over the visible wavelengths is very small, but shorter wavelengths do travel more slowly than do longer wavelengths.

SCATTERING

- The extinction cross section of an object consists of a part due to absorption and a part due to scattering
  \[ \sigma_\lambda = \sigma_\lambda^a + \sigma_\lambda^r. \]

- The scattering cross section can be written as
  \[ \sigma_\lambda^r = K A, \]
  where $A$ is the geometric (physical) cross-section of the object, and $K$ is the scattering area coefficient (or scattering efficiency).

- The scattering efficiency is the ratio of the scattering cross-section to the geometric gross section of the particle.

- The scattering efficiency will depend on the size of the particle, the wavelength of light, and on the index of refraction of the particle.

- We can combine the wavelength and particle size into one parameter called the size parameter, defined as $\alpha = 2\pi/\lambda$, because it is the ratio of the particle size to the wavelength of light that is important.

GEOMETRIC OPTICS REGIME
If the size parameter is greater than 50 or so, the $K$ approaches an asymptotic value of 2, and the scattering can be described using geometric optics.

In this regime, all wavelengths are scattered equally, so the scattered light will be of the same mixture of wavelengths as the incident light.

This is why clouds appear white.

**RAYLEIGH SCATTERING REGIME**

If the size parameter is small ($\alpha << 1$) then the scattering efficiency is given by

$$K = \frac{128}{6} \left( \frac{m_e^2 - 1}{m_e^2 + 1} \right) \alpha^4.$$ 

For a scatterer of a given size, this means that $K \propto \lambda^{-4}$, and that shorter wavelengths are scattered much more efficiently than are longer wavelengths.

For visible light, the ratio of $K_{\text{blue}}/K_{\text{red}}$ is about 3.5, so that blue light is scattered nearly 4 times more efficiently than is red light. This is why the sky appears blue.

**MIE SCATTERING REGIME**

In between the two extremes of the Rayleigh and geometric optics regimes, scattering is under the more complex Mie regime. In this regime, the scattering efficiency exhibits oscillatory behavior with increasing size parameter.

In the Mie regime, if the particles are nearly uniform in size, the scattered light may either appear blue or reddish, depending on whether $K$ is increasing or decreasing with $\alpha$.

If the particles are not uniform (i.e., there is a whole spectrum of sizes present) then several minima and maxima will be included, and the light will appear whitish.

**MIRAGES**

Mirages are caused by light bending due to changes in refractive index due to temperature changes along the path of the ray.
● Because light bends toward colder temperatures, if the air near the ground is very hot, the light rays will bend upward
  o This causes objects far away to appear inverted, and underneath their actual position
  o This results in an *inferior image*.
  o This is what causes hot asphalt to appear wet up ahead, because the sky appears as an inferior image in the road.
● If the air near the ground is very cold, the light rays bend downward
  o This causes far away objects to appear above their actual position
  o This results in a *superior image*
  o This can actually allow objects that are over the horizon to be seen.

**PRIMARY RAINBOWS**

● Rainbows are caused by the refraction and reflection of light by rain and cloud drops.
  o The index of refraction for water is much larger than that for air, and also depends on wavelength.
  o The index of refraction is greater for shorter wavelengths, so therefore, shorter wavelengths bend more.
● The figures below shows the path of a light ray through a spherical water droplet for a primary rainbow.

![Diagram of primary rainbow](image)

- The angle \( \theta_i \) is the incidence angle.
The interior angle between the incoming and outgoing ray, $\theta'$, is called the *bending angle*.

The exterior angle, $\theta''$, is called the *deviation angle*.

- Using Snell's Law and geometry, the angle $\theta'$ for the primary rainbow can be related to the incidence angle $\theta_i$ as follows (refer to diagram below).

\[
\frac{\theta_i}{\sin 1} = \arcsin \left( \frac{1}{m} \sin \theta_i \right)
\]

where we define the index of refraction of water with respect to air, $m$, as

\[
m \equiv m_{r(water)} / m_{r(air)}.\]

- Solving this system of equations for $\theta'$ gives

\[
\theta' = 4 \arcsin \left( \frac{1}{m} \sin \theta_i \right) - 2 \theta_i. \quad \text{Bending angle for primary rainbow} \quad (1)
\]

There are an infinite number of possible incidence angles, and therefore paths, through the droplet. The plot below shows the deviation and bending angles as a function of incidence angle.
Note that there is a certain incidence angle at which the deviation is a minimum, and therefore, the bending angle is a maximum. At this incidence angle, slight changes in the incidence angle \( \theta_i \) lead to no change in \( \theta' \).

At this special incidence angle, the light beam will be very concentrated. At other incidence angles, the light beam will not be as concentrated.

It is only at this special incidence angle, where the deviation angle is a minimum (and the bending angle is a maximum) that a rainbow will form.

To find this special incidence angle analytically, we need to differentiate equation (1) with respect to \( \theta_i \), set it equal to zero, and solve for \( \theta_i \) (this is left as an exercise). Doing so shows that for the primary rainbow,

\[
\cos^2 \theta_i = \left( m^2 - 1 \right) / 3. \quad \text{Incidence angle for primary rainbow} \quad (2)
\]

Equations (1) and (2) are used together to determine the angle that a primary rainbow subtends.

The value of \( m \) depends on the wavelength of the light. The table below shows these values for a few colors.

<table>
<thead>
<tr>
<th>Color</th>
<th>violet</th>
<th>green</th>
<th>orange</th>
<th>red</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda (\mu m) )</td>
<td>0.4047</td>
<td>0.5016</td>
<td>0.5893</td>
<td>0.7061</td>
</tr>
<tr>
<td>( m )</td>
<td>1.3427</td>
<td>1.3364</td>
<td>1.3330</td>
<td>1.3300</td>
</tr>
</tbody>
</table>

Since different colors have different indices of refraction, then each color will have a slightly different angle after it leaves the droplet. Thus, the droplets act as prisms to separate the colors.
- For red light, the angle $\theta'$ is approximately $42^\circ$. This is the angle that a primary rainbow subtends.
  - The shorter wavelengths have a smaller $\theta'$, because they have a larger index of refraction. Therefore, red appears as the outer band of a primary rainbow, with the other colors following in order of decreasing wavelength.
- If the Sun angle is larger than $42^\circ$, a primary rainbow cannot be seen by a ground-based observer.

**SECONDARY RAINBOWS**

- It is also possible to have a path through the water droplet that has two internal reflections, such as that shown below.

  ![Diagram of secondary rainbow](image)

  - For this case, $\theta'$ we have the following relations
    \[ \theta_i + \alpha = 180^\circ \]
    \[ (\theta'/2) + \beta + \alpha = 180^\circ \]
    \[ \beta + 3\gamma = 180^\circ \]
    \[ \gamma + \theta_i = 90^\circ \]
    \[ \theta_i = \arcsin \left( \frac{1}{m} \sin \theta_i \right) \]
  - Solving this system of equations for $\theta'$ gives
    \[ \theta' = 180^\circ + 2\theta_i - 6\arcsin \left( \frac{1}{m} \sin \theta_i \right). \quad \text{Bending angle for secondary rainbow} \quad (3) \]
- As before, there are an infinite number of such paths. However, there is one special path for which $d\theta'/d\theta_i = 0$, which gives an incidence angle of
  \[ \cos^2 \theta_i = \left( m^2 - 1 \right)/8. \quad \text{Incidence angle for secondary rainbow} \quad (4) \]
For red light, the angle $\theta'$ is approximately 50°. This is the angle that a secondary rainbow subtends.

The shorter wavelengths have a larger $\theta'$ in this case.

Therefore, red appears as the inner band of a secondary rainbow, with the other colors following in order of decreasing wavelength.

Secondary rainbows are much dimmer than primary rainbows, because every time the light refracts and/or reflects, some of the radiance is lost out of the beam.

If the Sun angle is larger than 50°, a secondary rainbow cannot be seen by a ground-based observer.

**PRISMS**

A prism is a transparent wedge of material which allows light to pass through it, and which separates the colors of the light.

There are many paths that the light can take through a prism. One such path is shown below.

For a prism, $\theta'$ is the *deviation angle*.

We can find the angle by which the light ray is bent through a prism by using the following relationships from the diagram,
\[ \alpha = \theta_i - \theta_i \]
\[ \beta = \theta_r - \theta_2 \]
\[ \gamma = 180^\circ - \alpha - \beta \]
\[ \theta' = 180^\circ - \gamma \]

from which we can deduce that
\[ \theta' = \theta_i + \theta_r - (\theta_i + \theta_2) . \]

From the diagram we can also see that
\[ \phi = 180^\circ - A = 180^\circ - (\theta_1 + \theta_2) \Rightarrow \theta_i + \theta_2 = A \tag{5} \]
so that
\[ \theta' = \theta_i + \theta_r - A . \tag{6} \]

- The figure below shows a plot of the deviation angle as a function of angle of incidence for a 70° prism with a refractive index of 1.2.

![Graph showing deviation angle vs. incident angle]

- As before, there are an infinite number of paths through the prism, but there is only one for which the beam will be concentrated (i.e., for which an infinitesimal change in \( \theta_i \) will produce no change in \( \theta' \)). This will occur when the deviation angle is a minimum (just like for the rainbow).

- We find this angle by taking
\[ \frac{d\theta'}{d\theta_i} = 0 . \]
Using equation (6) we get
\[ \frac{d\theta'}{d\theta_i} = 1 + \frac{d\theta_r}{d\theta_i} = 0 , \]
or

\[
\frac{d\theta_r}{d\theta_i} = -1. \tag{7}
\]

- From the chain rule and equation (7) we have

\[
\frac{d\theta_r}{d\theta_i} = \frac{d\theta_r}{d\theta_2} \cdot \frac{d\theta_2}{d\theta_1} \cdot \frac{d\theta_1}{d\theta_i} = -1. \tag{8}
\]

- From Snell’s law we know

\[
\frac{\sin \theta_i}{\sin \theta_i} = m \quad \frac{\sin \theta_r}{\sin \theta_2} = m
\]

which when differentiated gives

\[
\frac{d\theta_i}{d\theta_1} = m\frac{\cos \theta_i}{\cos \theta_i} \quad \frac{d\theta_r}{d\theta_2} = m\frac{\cos \theta_2}{\cos \theta_r}. \tag{9}
\]

Also, since \( \theta_i + \theta_2 = \pi \) (equation 5) we have

\[
\frac{d\theta_2}{d\theta_1} = -1. \tag{10}
\]

Using (9) and (10) in (8) yields

\[
\frac{\cos \theta_2 \cos \theta_i}{\cos \theta_1 \cos \theta_r} = 1. \tag{11}
\]

- The only way for condition (11) to be true is for the following to also be true:

\[
\cos \theta_i = \cos \theta_r\]

\[
\cos \theta_i = \cos \theta_2.
\]

This leads to the important conclusion that for a beam of light to pass through a prism without significant divergence of the light rays, the angle of incidence must equal the angle at which the beam leaves the prism.

- Another way of saying this is that **at the minimum deviation angle the beam passes symmetrically through the prism**, as shown below.
The deviation angle in this symmetric case (from 6) is
\[ \theta' = 2\theta_i - A. \] (12)

The angle of incidence is going to depend on the angle of the prism \((A)\) and the index of refraction, \(m\).

We know from equation (5) that
\[ \theta_i = A/2 \]
and from Snel’s Law that
\[ \frac{\sin \theta_i}{\sin \theta} = m, \]
which combined show that
\[ \sin \theta_i = m \sin(A/2), \] (13)
and therefore
\[ \theta' = 2 \arcsin[m \sin(A/2)] - A. \] Minimum deviation angle through a prism (14)

REFRACTION OF LIGHT THROUGH ICE CRYSTALS

- Ice crystals come in many different shapes, or habits.
  - The habit depends on the temperature at which the crystal formed.
- However, all ice crystals are six-sided (hexagonal).
- Optical phenomenon like halos and sun dogs are associated with columnar or plate-like ice crystals, like that shown below.
There are two possible symmetric paths of the light through the column, and these are shown below.

- One path is for the light to pass laterally across the crystal, such as shown on the left.
  - In this case, the crystal acts like a prism with an angle of $60^\circ$, and using the index of refraction for ice ($n = 1.31$) equation (14) gives an angle of bending of about $22^\circ$.
- The other possible path is for the light to pass longitudinally along the crystal, such as shown on the right.
  - In this case, the crystal acts like a prism with an angle of $90^\circ$, and using the index of refraction for ice ($n = 1.31$) equation (14) gives an angle of bending of about $46^\circ$. 
A third path (shown below) is impossible for ice crystals, because the index of refraction for ice is too large, and therefore, equation (13) gives a nonsensical result for \( \sin \theta_i \).

HALOS, SUN DOGS, AND SUN PILLARS
- Halos, sun dogs (also called perihelia), and sun pillars are caused by the reflection or refraction of sunlight from ice crystals.
- Halos and sun dogs are displaced either 22° or 46° from the Sun, depending on which path the light takes through the ice crystal (see below).

- Halos are arcs around the Sun, and occur when the ice crystals are randomly oriented.
- Sun dogs only appear to either or both sides of the sun, and occur when the ice crystals are primarily oriented vertically.
- Sometimes, halos and sun dogs appear together.
- Halos of other dimensions are sometimes observed, as well as tangent and circumzenithal arcs, and are associated with oddly shaped ice crystals which have a pyramidal cap, or to spinning ice crystal.
- Sun pillars are caused by reflection, rather than refraction, and appear as a column of light above (and sometimes below) the sun.
For sun pillars, the ice crystals are more plate-like, rather than columnar, and are oriented with the flat plate-like surfaces horizontally. They then act together like a large mirror.

The diagram below shows the various optical phenomena associated with ice crystals.

- a. are **sun dogs**
- b. is the 22° **halo**
- c. is the 46° **halo**
- d. are **sun pillars**
- e. is the **upper tangent arc**
- f. is the **circumzenithal arc**
- g. is the **lower tangent arc**
- h. is the **parhelic circle**
EXERCISES

1. Take $\frac{d}{d\theta}$ of equations (1) and (3), and set it equal to zero to derive equations (2) and (4) respectively.

2. You are on a planet where the rain drops are made up of a liquid with an index of refraction of 1.44. You notice a rainbow. Assuming it is a primary rainbow, what is the bending angle?

3. If ice crystals were pentagonal (5-sided) columns, list the angles of all possible haloes.

4. You are on a planet with strange clouds. You suspect the clouds are made up of some sort of crystal, and other measurements lead you to believe that the index of refraction of the crystal is 1.05. You have observed halos of $2.4^\circ$, $5.9^\circ$, and $16.9^\circ$. What would you guess the shape of the crystals are?

5. You are on a planet where the clouds are made up of droplets of an unknown liquid. You notice a rainbow (assume it is a primary rainbow) with an angle of $15.0^\circ$. What is the index of refraction (to two decimal places) of the cloud droplets?