The objectives of this lesson are:
1) Discuss the terminal velocity of droplets and its dependence on droplet size.
2) Derive an expression for the growth rate of a falling droplet through collision-coalescence.
3) Discuss the parameters that impact the collection efficiency.

TERMINAL VELOCITY
- The vertical forces on a falling droplet are:
  - Friction - $F_r = \frac{\pi}{2} r^2 u^2 \rho C_D$, where $C_D$ is the drag coefficient.
  - Gravity - $F_g = \frac{4}{3} \pi r^3 \rho_L g$

- Eventually the droplet will reach terminal velocity, at which point the two forces are in balance. This results in
  $$u^2 = \frac{8 \rho g \rho_L}{3 \rho C_D}.$$

- If we knew the drag coefficient, then we could find the terminal velocity. The drag coefficient is determined experimentally, with the following results obtained for the terminal velocity of droplets.

<table>
<thead>
<tr>
<th>Droplet Radius</th>
<th>Terminal Velocity</th>
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<tbody>
<tr>
<td>$r &lt; 40 \mu m$</td>
<td>$u = (1.19 \times 10^6 \text{ cm}^{-1} \text{ s}^{-1}) r^2$</td>
</tr>
<tr>
<td>$40 \mu m &lt; r &lt; 0.6 \text{ mm}$</td>
<td>$u = (8 \times 10^3 \text{ cm} \text{ s}^{-1}) r$</td>
</tr>
<tr>
<td>$r &gt; 0.6 \text{ mm}$</td>
<td>$u = (2.2 \times 10^3 \text{ cm}^{1/2} \text{ s}^{-1}) (\rho_0/\rho) r^{1/2}$</td>
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where $\rho_0$ is equal to 1.20 kg/m$^3$. 
GROWTH DUE TO COLLISION WITH SMALLER, UNIFORM DROPLETS

- As a droplet falls it may collide with other droplets.
- The volume swept out by a droplet of radius $R$ falling through a distance of $dh$ is $\pi R^2 dh$. However, if the droplet is falling through a population of smaller droplet each of radius $r$, then the ‘collection’ radius of the falling droplet is $R + r$ because it will touch any smaller droplet whose center is within a distance of $d = R + r$ from the center of the large droplet. So, the volume swept out by the droplet is (see figure)

$$dV = \pi (R + r)^2 \, dh.$$ (1)

- If the larger droplet is falling at its terminal velocity, $u(R)$, then $dh = u(R) \, dt$ and Eq. (1) becomes

$$dV = \pi (R + r)^2 \, u(R) \, dt.$$ (2)

- If we assume that as the large drop falls, any smaller droplets that are at least partially within the volume $dV$ will be ‘collected by’ or stick to the large drop. Therefore, its mass will increase by the liquid water content, $M$, of the smaller droplets, multiplied by the volume, $dV$. Thus,

$$dm = M \, dV = \pi (R + r)^2 \, M \, u(R) \, dt$$ (3)
• But the smaller droplets are also falling at their terminal velocity, \( u(r) \). We have to correct for this, because if the small droplets were falling at the same speed of the larger droplet there would be no increase in mass since there would be no collisions. The greater the difference in the terminal velocities, the fewer collisions and the less mass will be collected.

• The corrected formula is

\[
dm = M\Delta V = \pi (R + r)^2 M \left[ u(R) - u(r) \right] dt
\]  

\[ (4) \]

or, in terms of growth rate,

\[
\frac{dm}{dt} = \pi (R + r)^2 M \left[ u(R) - u(r) \right].
\]  

\[ (5) \]

• If all the smaller droplets within the volume coalesce (stick) to the falling droplet, then Eq. (5) could be used to predict the rate of mass growth of the falling droplet.

CORRECTION DUE TO COLLECTION EFFICIENCY

• In reality, not all the smaller droplets will collide with or stick (coalesce) to the larger droplet.

• One reason for this is that, even though a droplet may be in the path of the larger drop, the air flow around the larger drop may force the smaller droplet away as the drops pass. Think of a bug heading for your windshield. Sometimes the bug is forced up and over your car by the air flow, and is spared a gruesome fate.

• It is also possible for droplets that weren’t in the path of the larger droplet to be captured by the wake of the falling droplet. This is known as wake capture (think of leaves following behind a moving car in the autumn).

• Even if two droplets collide, they may not actually stick, or coalesce.
  
  o One reason for this is that there may be a microfilm of air between the droplets that prevents their surfaces from contacting.

• All the above effects are taken into account by defining a collection efficiency, \( E \), that is just the ratio of the number of small droplets that actually stick to the larger drop to the total number of small droplets within the volume.
The collection efficiency must be experimentally determined, and is going to depend on many physical factors such as the drop sizes involved, temperature, turbulence, wind shear, etc.

- Usually the collection efficiency is less than unity.

With collection efficiency taken into account, Eq. (5) for the growth of droplet mass becomes

\[ \frac{dm}{dt} = \pi EM(R + r)^2 [u(R) - u(r)]. \]  

Equation (6) is often written as

\[ \frac{dm}{dt} = K(R, r)M. \]  

where

\[ K(R, r) = \pi E(R + r)^2 [u(R) - u(r)] \]  

is called the gravitational collection kernel.

GROWTH EQUATION IN TERMS OF RADIUS

- In terms of the change in radius of the larger drop, the growth-rate equation is

\[ \frac{dR}{dt} = \frac{EM(R + r)^2}{4\rho_L R^2} [u(R) - u(r)]. \]  

- If we assume the smaller droplets have a negligible terminal velocity compared to the larger droplets, then we can use the approximations

\[ u(R) - u(r) \equiv u(R) \]
\[ R + r \equiv R \]

and write the growth equation in terms of radius as

\[ \frac{dR}{dt} = \frac{EM}{4\rho_L} u(R). \]  

- From the chain rule this can be written as

\[ \frac{dR}{dz} = \frac{dR}{dt} \frac{dt}{dz} = \frac{dR}{dz} \frac{dz}{dt} \frac{1}{dz}. \]

But \( \frac{dz}{dt} \) is the rate at which the drop is actually falling, and is the difference of the speed of any updrafts and the terminal velocity of the drop,
\[
\frac{\partial z}{\partial t} = U - u(R)
\]  

(12)

where \( U \) is the updraft velocity. Therefore, Eq. (10) becomes

\[
\frac{dR}{dz} = \frac{EM}{4\rho_l} \frac{u(R)}{U - u(R)}.
\]  

(13)

If the updraft velocity is negligible, then this becomes

\[
\frac{dR}{dz} = \frac{EM}{4\rho_l}.
\]  

(14)

- Equation (14) suggests that the droplet radius should decrease with height, which makes physical sense, since as the droplet falls (\( z \) getting smaller) its radius should be increasing through the collision/coalescence process.

GROWTH DUE TO COLLISION WITH SMALLER DROPLETS OF NON-UNIFORM SIZE

- The equations we’ve derived up to now apply only to a large droplet falling through a population of smaller droplets having a uniform size. In reality, the smaller droplets will not just have a single radius.
- If there are two different sizes of smaller drops, \( r_1 \) and \( r_2 \), then the growth rate in terms of mass will have a separate term for each smaller droplet size present and will be

\[
\frac{dm}{dt} = \pi E_1 M_1 (R + r_1)^2 [u(R) - u(r_1)] + \pi E_2 M_2 (R + r_2)^2 [u(R) - u(r_2)]
\]  

(15)

where \( M_1 \) and \( M_2 \) are the liquid water contents for drops having radius \( r_1 \) and \( r_2 \) respectively, and \( E_1 \) and \( E_2 \) are the collection efficiencies for drops having radius \( r_1 \) and \( r_2 \) respectively.
- In general, if there are \( N \) different sizes of little drops, then we have

\[
\frac{dm}{dt} = \pi \sum_{i=1}^{N} E_i M_i (R + r_i)^2 [u(R) - u(r_i)]
\]  

(16)

where each drop radius, \( r_i \), has its own collection efficiency and liquid water content, \( E_i \) and \( M_i \). Written in terms of the collection kernel, we have
\[
\frac{dm}{dt} = \sum_{i=1}^{N} K(R, r_i) M_i. \tag{17}
\]

- If the small droplets are distributed continuously then the summation becomes an integration,

\[
\frac{dm}{dt} = \int K(R, r) dM, \tag{18}
\]

and we know that

\[
dM = \frac{4}{3} \pi \rho_r r^3 n_d(r) dr, \tag{19}
\]

so that the growth equation for a continuous droplet spectrum is

\[
\frac{dm}{dt} = \frac{4}{3} \pi \rho_r \int_{0}^{\infty} K(R, r) r^3 n_d(r) dr. \tag{20}
\]

- Equation (20) must be solved numerically, and is complicated by the fact that the collection kernel depends on radius of the small droplets, so it must remain within the integral.
- The collection kernel is usually determined empirically.
- If we assume that \( R >> r \) and \( u(R) >> u(r) \) then we can show (see exercise 3) that

\[
\frac{dR}{dt} = \frac{EM}{4 \rho_L} u(R), \tag{21}
\]

and

\[
\frac{dR}{dz} = -\frac{EM}{4 \rho_L}, \tag{22}
\]

which are the same as Eqs. (10) and (13).

- Thus, regardless of whether or not our drop-size distribution is continuous, or if we just have one size of drops present, if we make the assumptions that \( R >> r \) and \( u(R) >> u(r) \) then Eqs. (10) and (13) are valid in either case.

**COMMENTS ON COLLISION-COALESCEENCE**

- The growth rate is very sensitive to the drop size spectrum, with broader spectra leading to faster growth rates.
- A broad spectrum is beneficial because it leads to greater relative velocity between the drops, and more chance for collision. In a narrow spectrum, all
the drops are falling at roughly the same velocity, and are less likely to collide.

- Clouds droplets are initially formed via diffusion, which leads to a narrowing, rather than a broadening of the drop size spectra.
- Computations using Eq. (20) with typical drop size spectra give growth rates too small to explain how precipitation sized drops form in reality.
- To achieve growth rates comparable to that observed in natural clouds (~15 minutes), a stochastic process is required, whereby a small population of fortunate drops happen to grow much faster than the average rate.
  - As these drops grow very large (4 to 5 mm), they become unstable and break apart. This creates some additional large drops which can themselves start to grow through collision-coalescence.
- The physical cause of these stochastic phenomena is a source of study, but turbulence is likely a factor.
- The collision-coalescence process is often called the warm-rain process, since it is the only way to explain precipitation formation in clouds that remain above freezing. However, it can also occur in cold clouds.
EXERCISES

1. A droplet of initial radius $R_0$ is at the base of a cloud ($z = 0$) having a liquid water content of $M$ and a steady updraft speed of $U$. The terminal velocity of the drop is given by $u = a R$ where $a$ is a constant.

   a. Show that the maximum height above the cloud base that the droplet will reach before it begins to fall is given by

   $$z_{\text{max}} = \frac{4 \rho_L}{EM} \left[ \frac{U}{a} \ln \left( \frac{U}{a R_0} \right) - \frac{U}{a} + R_0 \right].$$  
   (23)

   Hint: Integrate (11) from the base of the cloud to $z_{\text{max}}$. Also, note that at the top of the droplet’s trajectory the terminal velocity will be equal to the updraft speed.

   b. Integrate (10) to show that the time required for the droplet to reach the top of its trajectory is

   $$t_{\text{up}} = \frac{4 \rho_L}{EM a} \ln \left( \frac{U}{a R_0} \right).$$  
   (24)

   c. If the initial droplet radius is 40 $\mu$m in a cloud with a liquid water content of 1.5 g/m$^3$, and the updraft velocity is 2 m/s, what will be the radius of the droplet at the top of its trajectory?

   d. What is the maximum height that the droplet will reach, assuming collection efficiency of 1?

2. A droplet is at the very top of its trajectory in a cloud of liquid water content $M$ and updraft speed $U$. The terminal velocity of the drop is given by $u(R) = b \sqrt{R}$. Integrate (10) to show that the droplet radius as a function of time is

   $$R(t) = \left( \frac{U}{b} + \frac{EM b t}{8 \rho_L} \right)^2.$$
3. a. The terminal velocity, \( u(R) \), increases with increasing \( R \). The equation
\[
\frac{dR}{dz} = \frac{EM}{4\rho_L} \frac{u(R)}{U - u(R)}
\]
implies that for small droplets \( dR/dz \) will be positive, while for very large droplets \( dR/dz \) will be negative. What is the physical explanation for this?

b. When the terminal velocity of the droplet equals the updraft velocity the \( dR/dz \) becomes infinitely large! What is the physical significance of this?

4. a. Show that for a continuous spectrum of small drops that the growth equation in terms of the radius of the big drop is
\[
\frac{dR}{dt} = \frac{1}{3R^2} \int_0^R K(R,r) r^3 n_r(r) dr.
\]

b. Show that if we assume that \( R \gg r \), and \( u(R) \gg u(r) \), that the growth rate equation for a continuous small drop spectrum becomes
\[
\frac{dR}{dt} = \frac{\pi E u(R)}{3} \int_0^\infty r^3 n_r(r) dr
\]

c. Show that the expression above is equal to
\[
\frac{dR}{dt} = \frac{EM}{4\rho_L} u(R).
\]