ESCI 343 – Atmospheric Dynamics II
Lesson 11 - Rossby Waves

Reference: *An Introduction to Dynamic Meteorology (4rd edition)*, J.R. Holton
Atmosphere-Ocean Dynamics, A.E. Gill
Fundamentals of Atmospheric Physics, M.L. Salby

Reading: Holton, 7.7 and 12.3

BAROTROPIC ROSSBY WAVES

Rossby waves owe their existence to the principle of conservation of potential vorticity. We first start with a barotropic fluid, for which the principle of conservation of barotropic potential vorticity states that

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0\,.$$

Expanding this out yields the barotropic vorticity equation

$$\frac{\partial \zeta}{\partial t} + \vec{V} \bullet \nabla_h \zeta + \beta v = \frac{f}{h}\frac{Dh}{Dt}\,.$$

The linearized form of this equation with zonal mean flow only ($\bar{u} \neq 0; \bar{v} = 0$) is

$$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + \beta v' = \frac{f_0}{H}\left(\frac{\partial \eta}{\partial t} + \bar{u}\frac{\partial \eta}{\partial x} + \bar{u}\frac{\partial H}{\partial x} + u'\frac{\partial H}{\partial x} + v'\frac{\partial H}{\partial y}\right). \qquad \textbf{(1)}$$

If we assume that the mean depth of the fluid (*H*) is constant then equation (1) becomes

$$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + \beta v' = \frac{f_0}{H}\left(\frac{\partial \eta}{\partial t} + \bar{u}\frac{\partial \eta}{\partial x}\right). \qquad \textbf{(2)}$$

Further assuming geostrophic balance we can write them in terms of the perturbation height as follows,

$$\zeta' = \frac{g}{f_0}\nabla^2 \eta$$

$$v' = \frac{g}{f_0}\frac{\partial \eta}{\partial x}$$

so that equation (2) becomes

$$\frac{\partial}{\partial t}\left(\nabla^2\eta - \frac{f_0^2}{c^2}\eta\right) + \bar{u}\frac{\partial}{\partial x}\left(\nabla^2\eta - \frac{f_0^2}{c^2}\eta\right) + \beta\frac{\partial \eta}{\partial x} = 0 \qquad \textbf{(3)}$$

where *c* is the phase speed of a shallow-water gravity wave. The waves supported by equation (3) are called *Rossby waves*.

To find the dispersion relation for the waves supported by equation (3) we assume a perturbation of the form

$$\eta = Ae^{i(kx+ly-\omega t)}$$

and substitute it into (3). This yields the following dispersion relation for Rossby waves,

$$\omega = \bar{u}k - \frac{\beta k}{\mathrm{K}^2 + f_0^2/c^2}\,. \qquad \textbf{(4)}$$

For waves that are short compared to the Rossby radius of deformation (given by c/f_0), the wave number will be large compared to f_0/c. In this case the dispersion relation becomes

$$\omega = \bar{u}k - \frac{\beta k}{K^2}, \quad \textbf{(4')}$$[1]

a result known as the ***shortwave approximation***.

DISPERSION PROPERTIES OF ROSSBY WAVES

The phase velocity of Rossby waves with zero mean flow ($\bar{u} = 0$) is

$$\vec{c} = \frac{\omega}{K^2}\vec{K} = \frac{-k\beta}{K^2\left(K^2 + f_0^2/c^2\right)}\left(k\hat{i} + l\hat{j}\right), \quad \textbf{(5)}$$

and the group velocity is

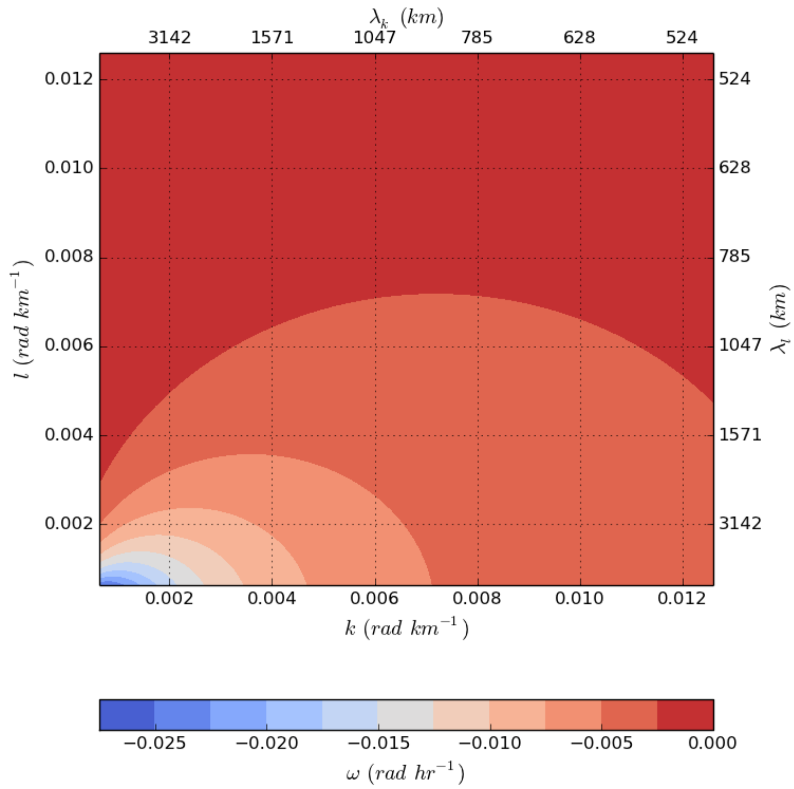
$$\vec{c}_g = \frac{\beta\left(k^2 - l^2 - f_0^2/c^2\right)}{\left(K^2 + f_0^2/c^2\right)^2}\hat{i} + \frac{2\beta kl}{\left(K + f_0^2/c^2\right)^2}\hat{j}. \quad \textbf{(6)}$$

The plots on the next page show the dispersion properties for Rossby waves with zero mean flow, using a value of $\beta = 10^{-11}$ m^{-1} s^{-1} and Rossby radius of deformation of 3000 km. Some things to note:

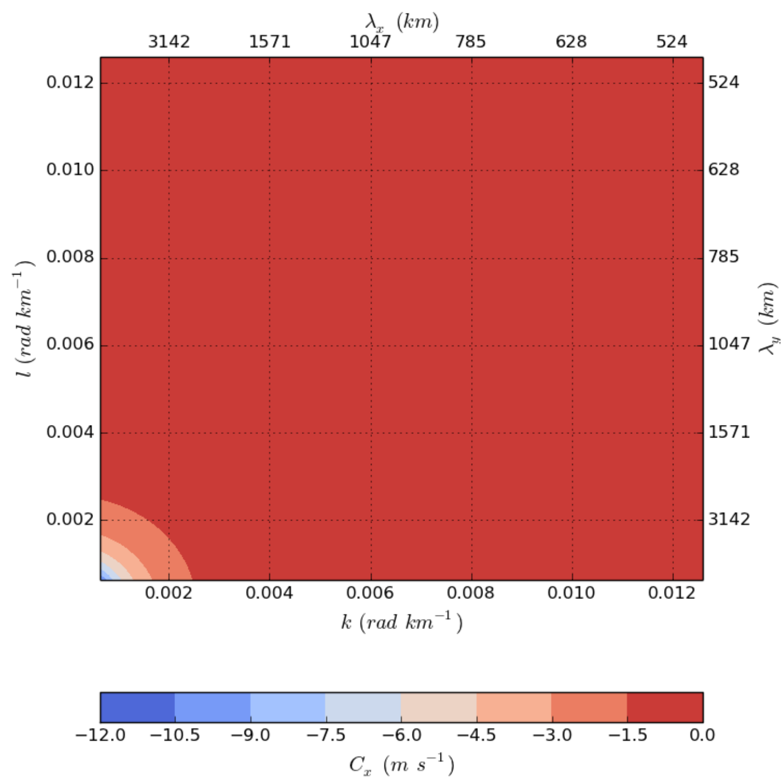
- The frequency is always negative, and becomes larger in magnitude for the longer wavelengths (smaller wave numbers).
- The zonal phase speed is always negative in the absence of mean flow.
- The zonal group speed may be either positive or negative, depending on the horizontal wave number.
    - Long waves propagate energy westward in the same direction as the phase speed.
    - Shortwaves propagate energy eastward, opposite to the phase speed.
- The meridional phase speed is negative, but the meridional group speed is positive.
    - The meridional energy propagation is opposite to the phase speed.

If there were a non-zero mean flow it would simply be added to the phase speeds and group speeds.

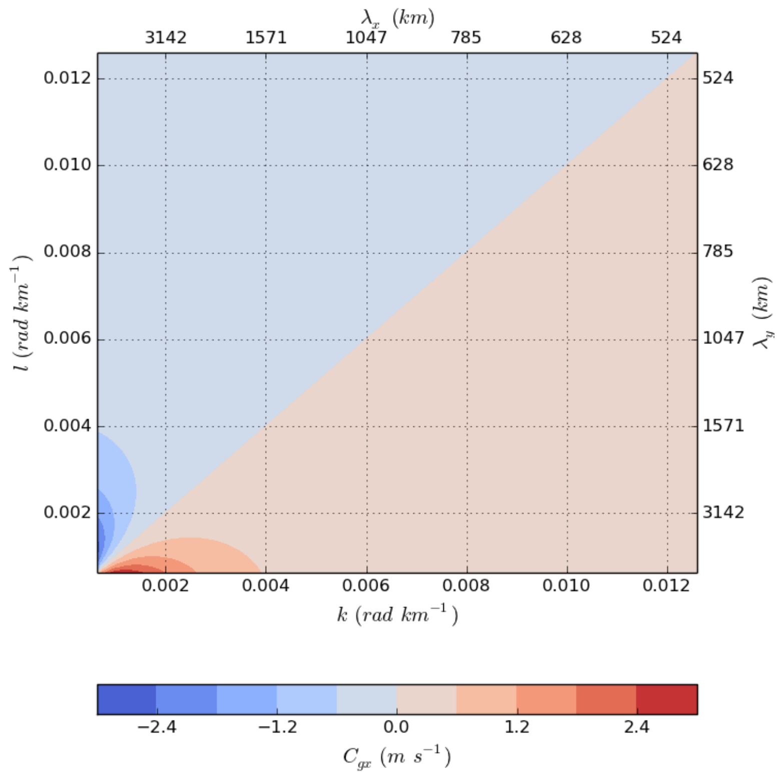
[1] In most meteorological textbooks equation (4') is the dispersion relation that is given for Rossby waves, and is derived directly by ignoring the right-hand-side of equation (2). Physically this implies ignoring the vertical stretching of the fluid column. Equation (4) is the more general form of the dispersion relation.



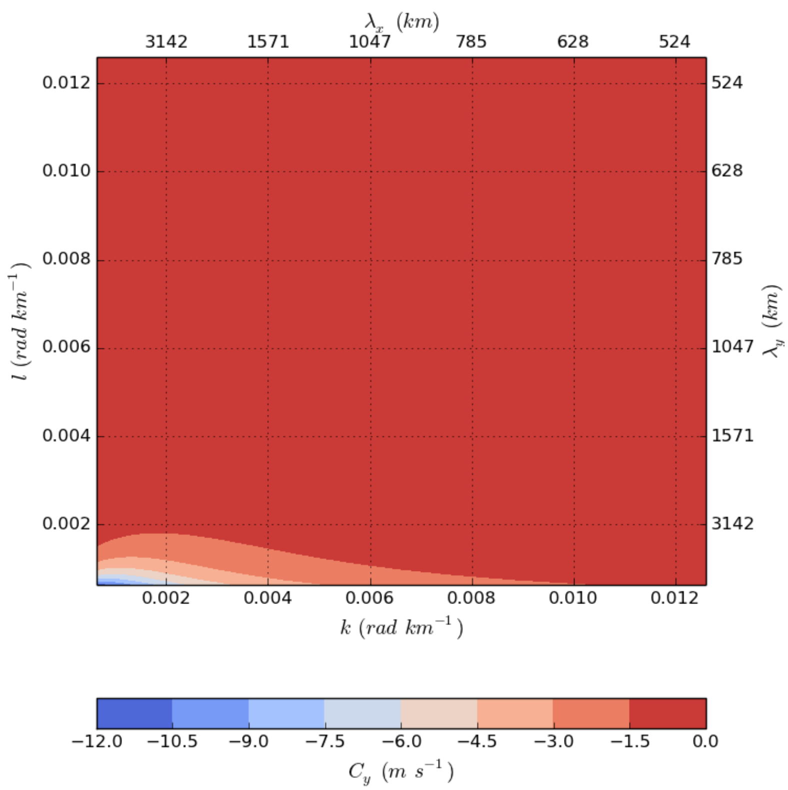
Angular frequency.



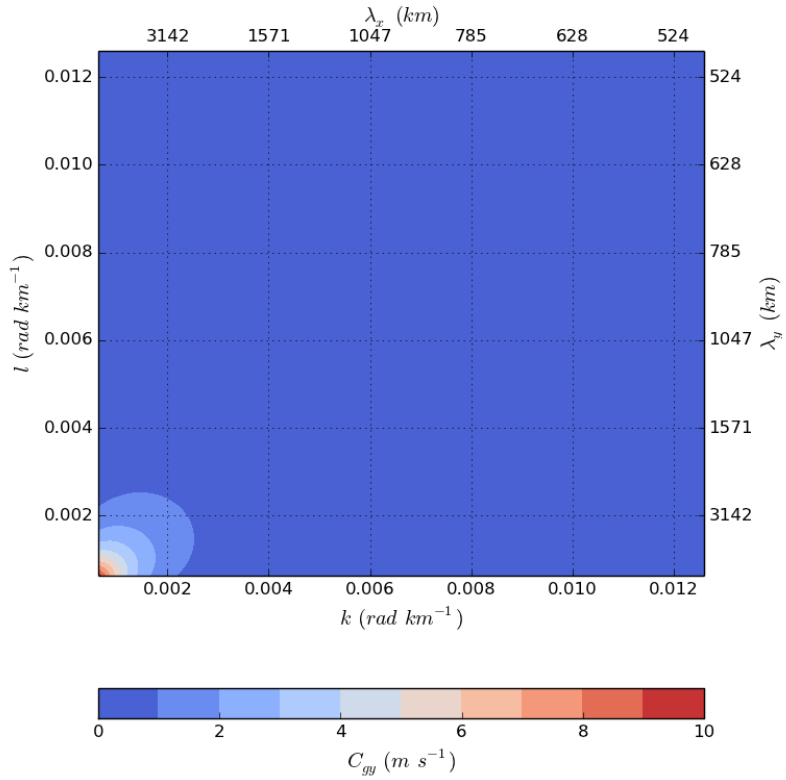
Zonal phase speed.



Zonal group speed.



Meridional phase speed.



Meridional group speed.

VERTICALLY PROPAGATING ROSSBY WAVES

In a stratified fluid it is possible to have Rossby waves that have a vertical component of propagation. To study these waves we have to use the concept of quasi-geostrophic potential vorticity (in place of the barotropic potential vorticity used in the previous discussions). In the absence of diabatic heating, quasi-geostrophic potential vorticity is conserved and therefore the following equation holds (see Lesson 4)

$$\frac{D_g}{Dt}\left[\frac{1}{f_0}\nabla^2\Phi + f + f_0\frac{\partial}{\partial p}\left(\frac{1}{\sigma}\frac{\partial\Phi}{\partial p}\right)\right] = 0$$

where the static-stability parameter is given as

$$\sigma = -\frac{\alpha}{\theta}\frac{\partial\theta}{dp}.$$

Converting this to height coordinates yields

$$\frac{D_g}{Dt}\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{f_0}{\bar{\rho}N^2}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial\Phi}{\partial z}\right)\right] = 0\,.$$

We can write this in terms of the streamfunction,

$$\psi \equiv \Phi/f_0\,,$$

so that

$$\frac{D_g}{Dt}\left[\nabla^2\psi + f + \frac{f_0^2}{\bar{\rho}N^2}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial\psi}{\partial z}\right)\right] = 0. \qquad \textbf{(7)}$$

The linearized form of equation (7), with mean flow in the zonal direction only, is

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + f + \frac{f_0^2}{\bar{\rho}N^2}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial\psi'}{\partial z}\right)\right] + \beta\frac{\partial\psi'}{\partial x} = 0. \qquad \textbf{(8)}$$

Since we are interested in waves which may propagate vertically great distances, we need to use a slightly modified form of the sinusoidal solution for the streamfunction,

$$\psi' = \frac{A}{\sqrt{\bar{\rho}}}e^{i(kx+ly+mz-\omega t)}, \qquad \textbf{(9)}$$

where $\bar{\rho}$ is a function of height. To simplify things we also assume and isothermal, anelastic atmosphere, so that

$$N^2 = -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz} = \frac{g}{H}.$$

Putting equation (9) into equation (8) yields the dispersion relation

$$\omega = \bar{u}k - \frac{\beta k}{K_H^2 + \left(\frac{f_0^2}{N^2}\right)\left(m^2 + \frac{N^4}{4g^2}\right)}. \qquad \textbf{(10)}$$

Rossby waves are often forced by topography. To find out when such stationary waves (with frequency zero) can propagate vertically we rearrange equation (10) for vertical wave number to get

$$m^2 = \left(\frac{N^2}{f_0^2}\right)\left[\frac{\beta}{\bar{u}} - K_H^2 - \frac{N^2 f_0^2}{4g^2}\right]. \qquad \textbf{(11)}$$

Vertical propagation is only possible if the term in brackets is positive. This means that for vertical propagation the mean zonal wind speed must be in the range

$$0 < \bar{u} \le \frac{\beta}{K_H^2 + N^2 f_0^2 / 4g^2}.$$

Therefore, ***Rossby waves cannot propagate vertically if the mean zonal winds are easterly, or if they are westerly and exceed a certain speed.***

This has important implications for the dynamics of the middle atmosphere (defined as the stratosphere and mesosphere). In the summertime the zonal winds in the middle atmosphere are easterly, and so energy from topographically forced Rossby waves cannot reach the middle atmosphere. In the wintertime, however, the zonal winds in the middle atmosphere are westerly, allowing Rossby waves to reach the middle atmosphere and deposit energy. This explains the sudden stratospheric warming episodes (as much as 40-50 K within a few days) observed in the Northern Hemisphere wintertime. This phenomenon is not as pronounced in the Southern Hemisphere because there are not as many topographical features in that hemisphere to generate topographically forced Rossby waves.

EXERCISES

1. For a mean zonal flow of 30 m/s, at what wavelength will a Rossby wave be stationary? Use β for 45°N, and assume $l = 0$.

2. Assume a barotropic fluid with a mean-depth, H, that varies in the y-direction only. Also, assume that $\beta = 0$, and that $\bar{v} = 0$.

a. Show that equation (1) then supports waves whose dispersion relation (using the short-wave approximation) is

$$\omega = \bar{u}k + \frac{f_0}{H}\frac{\partial H}{\partial y}\frac{k}{\mathrm{K}^2}\,.$$

These are Rossby waves that owe their existence to the bottom topography rather than to β, and can occur in the ocean along coastlines.

b. What is the phase speed and group velocity for these waves (assume $l = 0$)?

c. Assume that the bottom topography has an exponential shape such as $H(y) = A\exp(-\alpha y)$. What value should α be in order that these waves travel at the same speed as a regular Rossby wave at latitude 45°N?

3. a. Show for an isothermal atmosphere that

$$\bar{\rho}^{-1/2} = \rho_0^{-1/2} e^{z/(2H)}\,.$$

b. Use this result to show that equation (9) can be written as

$$\psi' = Ae^{i(kx+ly+[m+1/(2H)]z-\omega t)}\,.$$

c. Substitute this assumed solution into (8) to get the following dispersion relation for vertically propagating Rossby waves

$$\omega = \bar{u}k - \frac{\beta k}{\mathrm{K}_H^2 + \left(\frac{f_0^2}{N^2}\right)\left(m^2 + \frac{1}{4H^2}\right)}\,.$$

d. Show that this dispersion relation is identical to equation (10).