ESCI 342 – Atmospheric Dynamics I
Lesson 10 – Vertical Motion, Pressure Coordinates

Reading: Martin, Section 4.1

PRESSURE COORDINATES

- Pressure is often a convenient vertical coordinate to use in place of altitude.
- If the hydrostatic approximation is used, the relationship between pressure and altitude is given by the hydrostatic equation,

$$\frac{\partial p}{\partial z} = -\rho g \, . \tag{1}$$

- In height coordinates the vertical velocity is defined as $w \equiv Dz/Dt$. In pressure coordinates the vertical velocity is defined as

$$\omega \equiv \frac{Dp}{Dt} \, , \tag{2}$$

and is commonly called simply *omega*.
  - The units of ω are Pa/s (often microbars per second, μb/s, is also used).
  - ***Since pressure decreases upward, a negative omega means rising motion, while a positive omega means subsiding motion.***
- w and ω are related as follows:

$$\omega = \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} \, . \tag{3}$$

- On the synoptic scale we can assume hydrostatic vertical balance, so that (3) becomes

$$\omega = \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} - \rho g w \cong -\rho g w \, , \tag{4}$$

since the local pressure tendencies and horizontal pressure advection terms are much, much smaller in magnitude than the vertical pressure advection terms.
- The total derivative in pressure coordinates is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \, . \tag{5}$$ [1]

- The conversion of a height derivative to a pressure derivative is accomplished using the chain rule as follows

$$\frac{\partial}{\partial p} = \frac{\partial z}{\partial p}\frac{\partial}{\partial z} = -\frac{\alpha}{g}\frac{\partial}{\partial z} \, . \tag{6}$$

- In pressure coordinates, the directions of the unit vectors ($\hat{i}$, $\hat{j}$, and $\hat{k}$) are the same as in height coordinates. The x and y axes are still horizontal, and <u>not</u> oriented along the constant pressure surface.[2] The vertical axis is still vertical (perpendicular to x and y.)

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[1] If you take the total derivative of pressure you end up with the seeming absurdity that $\omega = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \omega$. However, the three partial-derivative terms are actually zero since they are taken holding pressure (the vertical coordinate) constant.

[2] See "The quasi-static equations of motion with pressure as independent variable", A. Eliassen, *Geof. Publ.*, **17**, 1949

- The u and v components of the wind are the same in both height and pressure coordinates.

MOMENTUM EQUATIONS IN PRESSURE COORDINATES

- In pressure coordinates the horizontal momentum equation is

$$\frac{D\vec{V}_H}{Dt} = -\nabla_p \Phi - \hat{k} \times f\vec{V}_H \qquad \textbf{(7)}$$

- The hydrostatic equation in pressure coordinates is

$$\frac{\partial \Phi}{\partial p} = -\alpha \,. \qquad \textbf{(8)}$$

CONTINUITY EQUATION IN PRESSURE COORDINATES

- The continuity equation is pressure coordinates is derived by writing the conservation of mass, *m*, for a parcel as follows:

$$\frac{Dm}{Dt} = \frac{D}{Dt}\left(\rho \delta x \delta y \delta z\right) = 0 \,. \qquad \textbf{(9)}$$

If the atmosphere is in hydrostatic balance, then $\rho \delta z = -\delta p / g$, so (9) becomes

$$\frac{D}{Dt}\left(\delta x \delta y \delta p\right) = 0 \,. \qquad \textbf{(10)}$$

Equation (10) expands out as

$$\frac{D}{Dt}\left(\delta x \delta y \delta p\right) = \delta y \delta p \frac{D}{Dt}\left(\delta x\right) + \delta x \delta p \frac{D}{Dt}\left(\delta y\right) + \delta x \delta y \frac{D}{Dt}\left(\delta p\right) = 0$$

which can also be written as

$$\frac{1}{\delta x}\frac{D}{Dt}\left(\delta x\right) + \frac{1}{\delta y}\frac{D}{Dt}\left(\delta y\right) + \frac{1}{\delta p}\frac{D}{Dt}\left(\delta p\right) = 0 \,. \qquad \textbf{(11)}$$

From the fact that

$$\frac{D}{Dt}\left(\delta x\right) = \delta u; \quad \frac{D}{Dt}\left(\delta y\right) = \delta v; \quad \frac{D}{Dt}\left(\delta p\right) = \delta \omega$$

equation (11) becomes

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$$

which in the limit as the parcel becomes infinitesimally small is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \,. \qquad \textbf{(12)}$$

- Equation (12) is the ***full continuity equation*** in pressure coordinates. It contains no assumptions about incompressibility.
- ***The full continuity equation in pressure coordinates looks very much like the incompressible continuity equation. This is one of the advantages of using pressure coordinates.***

THERMODYNAMIC ENERGY EQUATION IN PRESSURE COORDINATES

- The thermodynamic energy equation in pressure coordinates is

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J \ , \quad \textbf{(13)}$$

which expanded out, and using the definition of ω, becomes

$$\underset{A}{\frac{\partial T}{\partial t}} = \underset{B}{-\vec{V} \bullet \nabla_p T} - \left(\underset{C}{\frac{\partial T}{\partial p}} - \underset{D}{\frac{\alpha}{c_p}} \right) \omega + \underset{E}{\frac{J}{c_p}} \quad . \quad \textbf{(14)}$$

In this form, the terms represent:

Term A – Local temperature tendency
Term B – Horizontal thermal advection
Term C – Vertical thermal advection
Term D – Adiabatic expansion/compression due to vertical motion
Term E – Diabatic heating (radiation, latent heat, etc.)

- Terms *C* and *D* can be combined and written as

$$\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} = \left(\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p} \right) \frac{p}{R_d} \ ,$$

and defining the *static-stability parameter*, σ, as

$$\sigma \equiv -\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p} \ , \quad \textbf{(15)}$$

we get the following form of the thermodynamic energy equation in pressure coordinates.

$$\underset{A}{\frac{\partial T}{\partial t}} = \underset{B}{-\vec{V} \bullet \nabla_p T} + \underset{C}{\frac{\sigma p}{R_d} \omega} + \underset{D}{\frac{J}{c_p}} \ . \quad \textbf{(16)}$$

  - In this form of the equation, the vertical advection and adiabatic expansion/compression are combined into one term, Term C.
- The static stability parameter is a positive number for a stable atmosphere, and a negative number for an unstable atmosphere.

VERTICAL MOTION

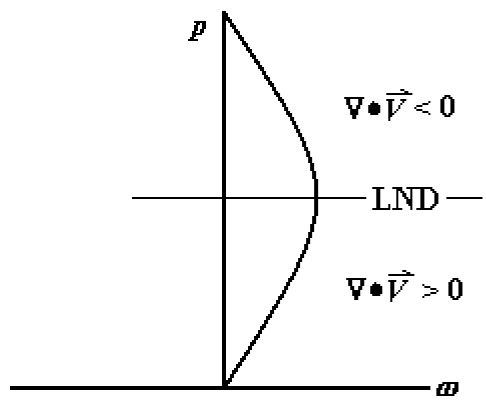
- Vertical motion is very important for forming clouds, and also effects the stability of the atmosphere; however, it is not routinely measured. Therefore, it must be inferred from other measured quantities.
- ***Kinematic method for calculating vertical motion***
  - One method of calculating the vertical motion is the *kinematic* method, which integrates the continuity equation between the surface and some pressure level above the surface to get

$$\omega(p) = \omega(p_s) + \int_p^{p_s} \nabla_p \bullet \vec{V} \, dp \ . \quad \textbf{(17)}$$

  - If the surface is flat and the surface pressure tendency is zero, then $\omega(p_s) = 0$ and (17) becomes

$$\omega(p) = \int_{p}^{p_s} \nabla_p \bullet \vec{V}\, dp . \qquad \textbf{(18)}$$

- o This gives the expected result that integrated convergence gives upward motion and integrated divergence gives downward motion.
- o The kinematic method has some major flaws. It is only the ageostrophic part of the wind field that can be divergent, and this is very small compared to the actual wind. In fact, the ageostrophic wind is of the order of the errors in the wind observations themselves. This means that divergences calculated from the observed winds may have large errors.
- o Though it isn't much use for calculating actual values of vertical motion, the kinematic method is good for illustrating some general points about divergence and its relation to vertical motion.
- o Since the vertical motion must disappear at the ground, and is also usually quite small at the top of the troposphere, a graph of the vertical motion with height would look something like that shown below



- o Since $\partial\omega/\partial p$ must disappear at some level, the divergence also disappears at that level. This leads to the conclusion that
  - ▪ ***There is some level in the atmosphere at which there is no horizontal divergence. This level is known as the level of non-divergence, or LND.***
- o Observations indicate that the level of non-divergence usually occurs at around 600 mb. However, since 600 mb is not a standard pressure level for reporting, traditionally meteorologists consider 500 mb to be the level of non-divergence.

- ***Adiabatic method for calculating vertical motion***
  - o Another method for calculating vertical motion uses the thermodynamic energy equation, (16), solved for ω and assuming adiabatic conditions, to get

$$\omega = \frac{R_d}{\sigma p}\left(\frac{\partial T}{\partial t} + \vec{V} \bullet \nabla_p T\right). \qquad \textbf{(19)}$$

  - o A drawback to this method is that temperature tendency must also be known.

PRESSURE TENDENCY EQUATION

- An equation for the change in pressure at a fixed point in the atmosphere can be derived as follows:
  - o Differentiate the hydrostatic equation with respect to time to get

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right) = -g\frac{\partial \rho}{\partial t}\ . \qquad \textbf{(20)}$$

Substituting for $\partial\rho/\partial t$ from the continuity equation gives

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) = g\nabla \bullet \left(\rho\vec{V}\right) = g\nabla_H \bullet \left(\rho\vec{V}\right) + g\frac{\partial}{\partial z}\left(\rho w\right)$$

which when integrated from some level *z* to the top of the atmosphere yields

$$\left.\frac{\partial p}{\partial t}\right|_{\infty} - \left.\frac{\partial p}{\partial t}\right|_{z} = g\int_z^{\infty}\nabla_H \bullet \left(\rho\vec{V}\right)dz - g\rho w\,. \qquad \textbf{(21)}$$

Since $\partial p/\partial t$ at the top of the atmosphere is zero, equation (21) for the pressure tendency at level *z* is

$$\frac{\partial p}{\partial t} = -g\int_z^{\infty}\nabla_H \bullet \left(\rho\vec{V}\right)dz + g\rho w$$

This equation can be expanded as

$$\frac{\partial p}{\partial t} = -g\int_z^{\infty}\rho\nabla_H \bullet \vec{V}\,dz - g\int_z^{\infty}\vec{V} \bullet \nabla_H\rho\,dz + g\rho w\,. \qquad \textbf{(22)}$$

Using the ideal gas law we can show that

$$\nabla_H\rho = R_d^{-1}\nabla_H\left(p/T\right) = R_d^{-1}T^{-2}\left[T\nabla_H p - p\nabla_H T\right]$$

so (22) becomes

$$\underset{A}{\frac{\partial p}{\partial t}} = \underset{B}{-g\int_z^{\infty}\rho\nabla_H \bullet \vec{V}dz} - \underset{C}{\frac{g}{R_d}\int_z^{\infty}\frac{1}{T}\left(\vec{V} \bullet \nabla_H p\right)dz} + \underset{D}{\frac{g}{R_d}\int_z^{\infty}\frac{p}{T^2}\left(\vec{V} \bullet \nabla_H T\right)dz} + \underset{E}{g\rho w}\ . \qquad \textbf{(23)}$$

- The physical interpretation of the pressure tendency equation is as follows:
  - ***Term A*** represents the local pressure tendency
  - ***Term B*** represents the vertically integrated divergence above the level of interest.
    - Integrated divergence above the layer leads to lower pressure.
    - Integrated convergence above the layer leads to higher pressure.
  - ***Term C*** represents integrated advection of pressure.
    - If the winds are in geostrophic or gradient balance, this term will be zero.
  - ***Term D*** represents the integrated temperature advection.
    - Advection of warm air lowers the pressure.
  - ***Term E*** represents advection of mass across the layer.
    - Upward vertical velocity leads to increased pressure, as the mass is moved above the level (since it is the mass above the level that determines the pressure in a hydrostatic atmosphere).
    - At the surface of the Earth, if the surface if level, then ***Term E*** would be zero.

EXERCISES

1. Show that the hydrostatic equation in pressure coordinates is $\partial\Phi/\partial p = -\alpha$. Hint: Start with $\partial p/\partial z = -\rho g$ and use the chain rule and the definition of geopotential.

2. Show that

$$\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} = \left(\frac{\alpha}{\theta}\frac{\partial \theta}{\partial p}\right)\frac{p}{R_d}.$$

Hint: Take $\partial/\partial p$ of $T = \theta\left(p/p_0\right)^{\kappa}$.

3. If horizontal advection and diabatic heating are negligible, then the local temperature tendency from the thermodynamic energy equation is

$$\frac{\partial T}{\partial t} = \frac{\sigma p}{R_d}\omega.$$

This equation says that if the atmosphere is stable then downward motion will result in an increase in temperature at a fixed level, while if the atmosphere is unstable then downward motion will result in a decrease in temperature at a fixed level. Give a ***physical*** explanation as to why this occurs.

4. Use the adiabatic method to estimate the 500 mb vertical velocity (ω) for the following situation. The temperature tendency is zero. The temperature at 600 mb is −13°C, at 500 mb it is −19°C. The wind at 500 mb is from the SW at 20 m/s, and the temperature at 500 mb increases toward the West at 1°C/100 km.

5. For a typical tropical cyclone (which is a warm-core circulation in gradient balance), explain whether each term in the pressure tendency equation contributes to surface development (lower pressures at the surface) or to weakening (higher pressures at the surface). Which terms do you think are most important for surface development?